Section 1-3, Mathematics 108

From 1-2

Rationalizing Denominators

I do not subscribe to the idea that a rationalized denominator is a better form.
Even so, you need to know how to do this sometimes.

Example:

$$\frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

More complicated example:

$$\frac{1}{\sqrt[3]{7}} = \frac{1}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7}}{\sqrt[3]{7}} \cdot \frac{\sqrt[3]{7}}{\sqrt[3]{7}} = \frac{\sqrt[3]{7} \cdot \sqrt[3]{7}}{\left(\sqrt[3]{7}\right)^3} = \frac{\sqrt[3]{49}}{7}$$

Scientific Notation

Using Scientific notation we represent real numbers as the product of 10 raised to an integer exponent times a number.

We can write this as $x = a \times 10^n$ where $a \in \mathbb{R}$ and $n \in \mathbb{Z}$
n is of course called the exponent and a is called the mantissa.
When $1 \le a \le 10$ we call this **normalized** scientific notation.

Every real number except zero can be normalized this way.

Write the number and then just move the decimal point after the first digits, keeping track of how many places you move it, which becomes the exponent.

$45.23 \to 4.523 \times 10^1$

$.00342 \to 3.42 \times 10^{-3}$

There are a few advantages to this representation of numbers when doing science.

- Very large and very small numbers can be represented compactly

- The accuracy or **significant figures** of a number is shows explicitly
  For example 1.3 and 1.300 are approximately equal but 1.300 is much accurate.
  The assumption is that only the last digit has error.

- Multiplication and Division is straight forward

  $$a \times 10^n \cdot b \times 10^m = ab \times 10^{n+m}$$
  $$\frac{a \times 10^n}{b \times 10^m} = \frac{a}{b} \times 10^{n-m}$$

  The resulting mantissa ab or $\frac{a}{b}$ might not be between 1 and 10 so you need to **normalize** the number, that is move the decimal place.

  Example: $5.2 \times 10^3 \cdot 7.1 \times 10^{-6} = 36.92 \times 10^{-3} = 3.692 \times 10^{-4} = 3.7 \times 10^{-4}$

  In the last step we round because we can't end up with a result more accurate than the input values.

Note: Adding and Subtracting is tricky because you may need to de-normalize the numbers to align their decimal points.

Section 1-3

Algebraic Expressions

A **monomial** is an expression of the form ax^n where a is a constant, x is a variable, and n is a natural number. n is sometimes called the **degree** of the polynomial.

A **polynomial** is an expression formed from the sum of monomials, eg.

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

Each monomial in the expression is called a **term.**

The a_i 's are constants, with a_0 called the **constant term.**

n the largest exponent of the variable is **degree** of the polynomial.

You should already be familiar with first degree polynomials such as

$$5x + 7$$

and second degree polynomials such as

$$3x^2 - 2x + 8$$

Adding Polynomials

When adding and subtracting polynomials, we match up **like** terms, which are terms with the same degree.

$$\left(5x^4 + 3x^2 - 2x + 1\right) + \left(4x^3 + x^2 + 3x + 5\right) =$$

$$\begin{array}{|ccccc} 5x^4 & & 3x^2 & -2x & 1 \\ + & 4x^3 & x^2 & 3x & 5 \\ 5x^4 & 4x^3 & 4x^2 & x & 6 \end{array}$$

Subtracting Polynomials

To multiply polynomials we use the distributive law.

$$\left(3x^2+1\right)\left(x^3+x\right)=3x^2\left(x^3+x\right)+1\left(x^3+x\right)=3x^5+3x^3+x^3+x=3x^5+4x^3+x$$

You can do this type of multiplication the same way you would do multiplication:

$$\begin{array}{|ccccc}
 & & 2x^2 & -x & 4 \\
 & \times & & 2x & -3 \\
 & & -6x^2 & 3x & -12 \\
+ & 4x^3 & -2x^2 & 8x & \\
 & 4x^3 & -8x^2 & 11x & -12
\end{array}$$

Factoring Polynomials

Just as we can factor integers, eg.

$48=2\times2\times2\times2\times3=2^4\times3$ (Note these are prime factors

Polynomials can be factored, eg.

$$x^2+5x+6=\left(x+3\right)\left(x+2\right)$$

Important Patterns

There are a number of ways of factoring polynomials. To start with, there are some important patterns that you should be familiar with:

$$A^2+2AB+B^2=\left(A+B\right)^2$$
$$A^2-2AB+B^2=\left(A-B\right)^2$$
$$A^2-B^2=\left(A+B\right)\left(A-B\right)$$

For example:

$$4x^2+12x+9=\left(2x\right)^2+2x\cdot3+\left(3\right)^2=\left(2x+3\right)^2$$

or

$$9x^2-1=\left(3x\right)^2-1^2=\left(3x+1\right)\left(3x-1\right)$$

Polynomials of the form $A^2 + B^2$ are **irreducible**, meaning they can't be factored.

Well, that's not quite true. They can't be factored using just real numbers. This will change when we introduce complex numbers later.

Other Patterns

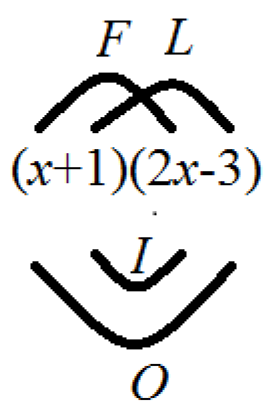
$$A^3 + 3A^2B + 3AB^2 + B^3 = (A+B)^3$$
$$A^3 - 3A^2B + 3AB^2 - B^3 = (A-B)^3$$
$$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$$
$$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$$

Factoring 2nd degree polynomials

Your first instinct to factor a 2nd degree polynomial should be using **reverse foil**.

Foil is a way to multiply two first degree polynomials.

Foil stands for First, Outside, Inside and Last
$(x+1)(2x-3) = [F]2x^2 - [O]3x + [I]2x - [L]3 = 2x^2 - x - 3$



To do reverse foil, break write a potential factoring

$x^2 - 5x + 6 = (x?2)(x?3)$ and see what signs, + or - will make it work by using foil.
In this case we see that two negative signs will work

$(x-2)(x-3) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$

When dealing with constants with more than one set of factors, trial and error can take some time.

Using roots to factor polynomials

If we set a polynomial equal to zero, we have a polynomial equation.

The places where the polynomial equals zero are called the **roots** of the equation.

By finding the roots we also can find the factors of the polynomial.

If for example a 2nd degree polynomial has roots 3 and -5 we know that the polynomial factors to:

$$(x-3)(x-(-5))$$

How do we know this? It's obvious. If you plug in 3 or -5 to this expression you obviously get zero, and if you plug 3 or -5 into the polynomial you also get zero because they are the roots, so the expression must be equivalent.

Finding roots of a polynomial equation

So how do we find the roots of a polynomial equation. If you can factor the polynomial using foil you are immediately done. What if this does not seem possible? The next technique is called **completing the square.** It uses the pattern

$$A^2 \pm 2AB + B^2 = (A \pm B)^2$$

Example:

$$x^2 + 6x + 7 = 0$$

We see that A=1 and 2AB=6x so B=3. Then $3^2 = 9$ is what completes the square.

$$x^2 + 6x + 7 = x^2 + 6x + 9 - 2 = (x+3)^2 - 2 = 0$$

So we have $(x+3)^2 = 2$

We find the square root of both sides, noting that the square root of a number can be positive or negative:

$$\sqrt{(x+3)^2} = \pm\sqrt{2} \quad x+3 = \pm\sqrt{2} \quad x = -3+\sqrt{2}, -3-\sqrt{2}$$

So the factors are $(x+3-\sqrt{2})(x+3+\sqrt{2})$

Using the Quadratic Formula to find roots

Of course we could use this technique on $Ax^2+Bx+C=0$ once and for all

$$Ax^2+Bx+C=0$$

Divide by A

$$x^2+\frac{B}{A}x+\frac{C}{A}=0$$

Complete the square

$$x^2+\frac{B}{A}x+\left(\frac{B}{2A}\right)^2+\frac{C}{A}-\left(\frac{B}{2A}\right)^2=0$$

Factor the perfect square

$$\left(x+\frac{B}{2A}\right)^2+\frac{C}{A}-\left(\frac{B}{2A}\right)^2=0$$

Moving the constants to the right side and finding the square root:

$$\sqrt{\left(x+\frac{B}{2A}\right)^2}=\pm\sqrt{\left(\frac{B}{2A}\right)^2-\frac{C}{A}}$$

we get

$$x+\frac{B}{2A}=\pm\sqrt{\left(\frac{B}{2A}\right)^2-\frac{C}{A}}$$

or

$$x=-\frac{B}{2A}\pm\sqrt{\left(\frac{B}{2A}\right)^2-\frac{C}{A}}=-\frac{B}{2A}\pm\sqrt{\frac{B^2-4AC}{4A^2}}=-\frac{B}{2A}\pm\frac{\sqrt{B^2-4AC}}{2A}=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

Giving us the familiar **Quadratic formula**

$$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$$

Example:

$$x^2+6x+7=x^2+6x+9-2=(x+3)^2-2=0$$

$$x=\frac{-6\pm\sqrt{36-28}}{2}=\frac{-6\pm\sqrt{8}}{2}=\frac{-6\pm 2\sqrt{2}}{2}=-3\pm\sqrt{2}$$

So the roots are $-3+\sqrt{2}\ and\ -3-\sqrt{2}$

and the factors are $\left(x+3-\sqrt{2}\right)\left(x+3+\sqrt{2}\right)$

Another advantage of the Quadratic formula is that it will continue to work when learn about complex numbers. Some polynomials, eg. x^2+A^2 can only be factored using complex numbers.

Factoring by Grouping

Sometimes a 3rd degree polynomial or higher will succumb to factoring by grouping. Here is an example:

$$x^4+x^2+4x+4$$

We group the first two terms and the last two and factor each separately.

$$x^3+x^2+4x+4=\left(x^3+x^2\right)+\left(4x+4\right)=x^2\left(x+1\right)+4\left(x+1\right)$$

Notice that the two groups have a common factor $(x+1)$, which can be factored out using the distributive law.

$$x^2\left(x+1\right)+4\left(x+1\right)=\left(x+1\right)\left(x^2+4\right)$$

Using the rational root theorem.

The rational root theorem says that if you have a polynomial of the form

$$a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

and it has a rational root, then that root must be of the form:

$\pm\frac{b_0}{b_n}$ where b_0 is a factor of a_0 and b_n is a factor of a_n

The strategy is then to find all the possible values $\pm\frac{b_0}{b_n}$, plug them into the polynomial and see which of them are roots. If you find a root *r* then you can divide the polynomial by $(x-r)$ and repeat the process until you either get a 2nd degree polynomial. Then either the polynomial will be irreducible or will fall to the quadratic formula.

Example:

$$x^4 - 3x^3 - x + 3$$

The possible roots are $1, -1, 3, -3$

Trying -3 we find that $(3)^4 - 3(3)^3 - (3) + 3 = 81 - 81 + 3 - 3 = 0$ so 3 is a root.
Dividing by $(x-3)$ we get

$$x^4 - 3x^3 - x + 3 = (x-3)(x^3 - 1)$$

Now using the special pattern $(A^3 - B^3)$ we see that

$(x-3)(x^3-1) = (x-3)(x-1)(x^2+x+1)$ so 1 is also a root.

Since $\frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1 \pm \sqrt{-3}}{2}$ is complex we are done.

Supplemental (You don't need to know this)

Since there is a quadratic formula, is there a similar solution for polynomial equations of the third degree, a cubic equation?

Yes!

Niccolò Fontana Tartaglia, an Italian mathematician who lived from 1499-1557 came up with this formula.

Given the equation $ax^3 + bx^2 + cx + d = 0$

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$- \frac{b}{3a}$$

Similarly there is a quartic formula discovered by Lodovico Ferrari in 1540.

The search for a formulaic solution to the general 5th degree equation went on for almost 300 years until Niels Henrik Abel, a Norwegian mathematican showed in 1820 that no such formula could exist.

The search for this formula served to develop what is known today as the subject "Modern Algebra", a course you might take as an undergraduate mathematics major after Calculus and Linear Algebra.